THE WEIL ALGEBRA OF A HOPF ALGEBRA

I - A noncommutative framework

Michel DUBOIS-VIOLETTE [1] and Giovanni LANDI [2]

Abstract

We generalize the notion, introduced by Henri Cartan, of an operation of a Lie algebra $\mathfrak{g}$ in a graded differential algebra Ω. Firstly we construct a natural extension of the above notion from $\mathfrak{g}$ to its universal enveloping algebra $U(\mathfrak{g})$ by defining the corresponding operation of $U(\mathfrak{g})$ in Ω. We analyse the properties of this extension and we define more generally the notion of an operation of a Hopf algebra $\mathcal{H}$ in a graded differential algebra Ω which is refered to as a $\mathcal{H}$-operation. We then generalize for such an operation the notion of algebraic connection. Finally we discuss the corresponding noncommutative version of the Weil algebra: The Weil algebra $W(\mathcal{H})$ of the Hopf algebra $\mathcal{H}$ is the universal initial object of the category of $\mathcal{H}$-operations with connections.

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[1]Laboratoire de Physique Théorique, UMR 8627
Université Paris XI, Bâtiment 210
F-91 405 Orsay Cedex
Michel.Dubois-Violette@u-psud.fr

[2]Dipartimento di Matematica, Università di Trieste
Via A. Valerio 12/1, I-34127 Trieste, Italy
and INFN, Sezione di Trieste, Trieste, Italy
landi@univ.trieste.it

Contents

1 Introduction

This paper is devoted to the noncommutative version of the notion of Cartan operation, [7], [8]. We first extend the notion of an operation of a Lie algebra $\mathfrak{g}$ in a graded differential algebra Ω as the notion of an operation of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ in Ω. It turns out that the properties of an operation of $U(\mathfrak{g})$ in Ω do only depend on the Hopf algebra structure of $U(\mathfrak{g})$. Since these properties still make sense if one replaces $U(\mathfrak{g})$ by an arbitrary Hopf algebra, this allows us to define the notion of an operation of a Hopf algebra $\mathcal{H}$ in a graded differential algebra Ω. We observe that various extensions of the Cartan calculus for the differential calculi on quantum groups (see e.g. [29] and [2]) fall in this framework. We then introduce the notion of an algebraic connection for an operation of a Hopf algebra $\mathcal{H}$ in a graded differential algebra Ω generalizing thereby the corresponding notion induced by Henri Cartan [7] (see also in [25]) for an operation of a Lie algebra in a graded <u>commutative</u> differential algebra.
The category of operations with connections of a given Hopf algebra $\mathcal{H}$ in graded differential algebras has a universal initial object $W(\mathcal{H})$ which we describe and which is the appropriate generalization of the Weil algebra. The relation with the construction of [18] is pointed out. It should be stresses that $W(\mathcal{H})$ is not connected with the noncommutative Weil algebra of [1], even when $\mathcal{H} = U(\mathfrak{g})$. However the purpose of the nice construction of [1] is different from that of this paper.

The plan of the paper is the following.
In Section 2 we summarize the elements of the classical theory that will be generalized in this paper. In this summary we have deliberately dropped two assumptions in the definition of an operation of a Lie algebra $\mathfrak{g}$ in a graded differential algebra Ω : The first one is the axiom $(i_X)^2 = 0$, $\forall X \in \mathfrak{g}$ and the second one is the graded commutativity of Ω. Indeed, as explained in the conclusion $(i_X)^2 = 0$, $\forall X \in \mathfrak{g}$ follows from the other axioms in all cases of interest and plays no role otherwise while the operation of $\mathfrak{g}$ in a noncommutative Ω makes sense and is useful as pointed out for instance in [20] for the case where $\mathfrak{g}$ is the Lie algebra $\mathrm{Der}(\mathcal{A})$ of all derivations of an algebra $\mathcal{A}$ and where Ω is the universal differential calculus $\Omega(\mathcal{A})$ over $\mathcal{A}$, [10], [11],[27], [28].

At this point, it is worth noticing that the classical definition of connections, of the Weil algebra, etc. are restricted to the frame of graded

commutative differential algebras. Our aim in the next sections is to generalize this classical theory to the noncommutative setting in two respects: firstly to replace the Lie algebra $\mathfrak{g}$, or more precisely its universal enveloping algebra $U(\mathfrak{g})$, by an arbitrary Hopf algebra $\mathcal{H}$ and secondly to allow arbitrary (noncommutative) graded differential algebras.

In Section 3, starting from an operation of a Lie algebra $\mathfrak{g}$ in a graded differential algebra $X \mapsto i_X$ we define its extension $h \mapsto i_h$ for h in the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. We formulate this extension in terms of the Hopf algebra structure of $U(\mathfrak{g})$.

In Section 4, we define more generally the notion of operation of a Hopf algebra $\mathcal{H}$ in a graded differential algebra Ω.

Section 5 is devoted to the noncommutative generalization of algebraic connections and to the description of an important example for the sequel.

In Section 6 we define the Weil algebra $W(\mathcal{H})$ of a Hopf algebra as the universal initial object of the category of $\mathcal{H}$-operations with connections.

Throughout this paper $\mathbb{K}$ denotes a field and all vector spaces and algebras are over $\mathbb{K}$. By an algebra (resp. a Lie algebra) without other specification we always mean a unital associative algebra (resp. a finite-dimensional Lie algebra) ; the unit of such an algebra will be denoted by $\mathbb{1}$ whenever no confusion arises. Except in §4.5 where $\mathbb{Z}$-graduations are considered, by a graded algebra, we mean a $\mathbb{N}$-graded algebra $\mathcal{A} = \oplus_{n\geqslant 0}\mathcal{A}_n$. Given a vector space E, its dual is denoted by E^* and given a linear mapping $\varphi : E \to F$ we denote by $\varphi^t : F^* \to E^*$ the corresponding transposed linear mapping. The tensor algebra of E is denoted by $T(E) = \oplus_n E^{\otimes^n}$, the symmetric algebra of E is denoted by $SE = \oplus_n S^n E$ and the exterior algebra of E is denoted by $\wedge E = \oplus_n \wedge^n E$. If E, F, G, H are $\mathbb{Z}$-graded vector spaces and if $\varphi : E \to F$ and $\psi : G \to H$ are homogeneous linear mappings with ψ of degree p, then their tensor product $\varphi \otimes \psi : E \otimes G \to F \otimes H$ is defined by (Koszul rule)

$$(\varphi \otimes \psi)(\alpha \otimes \beta) = (-1)^{pa}\varphi(\alpha) \otimes \psi(\beta)$$

for $\alpha \in E$ homogeneous of degree a and $\beta \in G$. We use everywhere the Einstein summation convention of repeated up-down indices in the formulas.

2 The classical theory

We review in this section some basic definitions and facts on the notion of operation introduced by Henri Cartan in 1950, [7], [8], (see also [25]).

2.1 Definition

Let $\mathfrak{g}$ be a Lie algebra and let Ω be a graded differential algebra with differential denoted by d.

An *operation of the Lie algebra* $\mathfrak{g}$ *in the graded differential algebra* Ω is a linear mapping

$$X \mapsto i_X \tag{2.1}$$

of $\mathfrak{g}$ into the space $\mathrm{Der}^{-1}(\Omega)$ of the antiderivations (graded-derivations) of degree -1 of Ω such that if one defines the derivation L_X of degree 0 by

$$L_X = i_X d + d i_X \tag{2.2}$$

for $X \in \mathfrak{g}$, then one has

$$[i_X, L_Y] = i_{[X,Y]} \tag{2.3}$$

for any $X, Y \in \mathfrak{g}$.

It follows from (2.2) and (2.3) that one has

$$[L_X, d] = 0 \tag{2.4}$$

and

$$[L_X, L_Y] = L_{[X,Y]} \tag{2.5}$$

for any $X, Y \in \mathfrak{g}$. Relation (2.5) means that one has a homomorphism of Lie algebras L of $\mathfrak{g}$ into the Lie algebra $\mathrm{Der}^0(\Omega)$ of all derivations of degree 0 of Ω.

2.2 Invariance, horizontality and basicity

An element α of Ω is said to be *invariant* if one has

$$L_X(\alpha) = 0 \tag{2.6}$$

for any $X \in \mathfrak{g}$ while α is said to be *horizontal* if one has

$$i_X(\alpha) = 0 \tag{2.7}$$

for any $X \in \mathfrak{g}$. Finally, $\alpha \in \Omega$ is said to be *basic* if it is both invariant and horizontal, that is if one has

$$L_X(\alpha) = i_X(\alpha) = 0 \tag{2.8}$$

for all $X \in \mathfrak{g}$.

The set Ω_I of all invariant elements of Ω is a graded differential subalgebra of Ω, the set Ω_B of all basic elements of Ω is a graded differential subalgebra of Ω_I (and therefore of Ω) while the set Ω_H of all horizontal elements of Ω is only a graded subalgebra of Ω which is stable by the L_X ($X \in \mathfrak{g}$). The cohomology $H_I(\Omega)$ of Ω_I and the cohomology $H_B(\Omega)$ of Ω_B are refered to respectively as the *invariant cohomology* and the *basic cohomology* of Ω (whenever no confusion arises concerning the operation).

2.3 The standard example

The whole terminology above comes from the theory of differential forms on principal bundles. Let G be a Lie group with Lie algebra $\mathrm{Lie}(G) = \mathfrak{g}$ and let $P = P(M, G)$ be a principal G-bundle over M (the basis) [30] with projection $\pi : P \to M$. The projection π induces a projection $T(\pi) : T(P) \to T(M)$ of the tangent bundle of P onto the tangent bundle of M and induces by duality an injective homomorphism $\pi^* : \Omega(M) \to \Omega(P)$ of graded differential algebras of the space $\Omega(M)$ of differential forms on the basis M into the space $\Omega(P)$ of differential forms on P. The image $\pi^*(\Omega(M))$ of π^* is denoted by $\Omega_B(P)$ and its elements are called basic differential forms on P. A tangent vector to P is said to be vertical whenever its projection on $T(M)$ via $T(\pi)$ vanishes. To each $X \in \mathfrak{g}$ corresponds a fundamental vector field on P which is vertical and which we also denote by X. The horizontal forms on P are the forms on P such that the inner derivations with the vertical vector fields vanish. Finally a form on P invariant by the action of the structure group is said to be invariant. Let i_X denote the inner derivation of $\Omega(P)$ by the fundamental vector field corresponding to $X \in \mathfrak{g}$. Then one verifies that $X \mapsto i_X$ defines an operation of $\mathfrak{g}$ in $\Omega(P)$ and that the notions of basicity, horizontality and invariance for the elements of $\Omega(P)$ correspond to the ones

associated to this operation.
It is worth noticing here that the notion of Cartan operation appears in some related example, for instance it plays a fundamental role in the computation of the local BRS cohomology of gauge theory, [24], [19], [22].

2.4 Algebraic connections for $\mathfrak{g}$-operations

In the sequel of this section we shall be concerned with operations of a Lie algebra $\mathfrak{g}$ in graded differential algebras which are graded commutative, they will be refered to as $\mathfrak{g}$-*operations*.

Given such an operation of a Lie algebra $\mathfrak{g}$ in a graded commutative differential algebra Ω *an algebraic connection* or simply *a connection* in Ω is a linear mapping

$$\alpha : \mathfrak{g}^* \to \Omega^1 \tag{2.9}$$

of the dual vector $\mathfrak{g}^*$ of $\mathfrak{g}$ such that

$$i_X(\alpha(\theta)) = \theta(X) \tag{2.10}$$

and

$$L_X(\alpha(\theta)) = \alpha(\theta \circ \mathrm{ad}(X)) \tag{2.11}$$

for any $X \in \mathfrak{g}$ and $\theta \in \mathfrak{g}^*$.

By the universal property of the exterior algebra, α extends as an homomorphism again denoted by

$$\alpha : \wedge\mathfrak{g}^* \to \Omega \tag{2.12}$$

of graded commutative algebras.

In fact $\wedge\mathfrak{g}^*$ is a graded differential algebra (endowed with the Koszul differential d) and the *curvature of* α is the linear mapping

$$\varphi : \mathfrak{g}^* \to \Omega^2 \tag{2.13}$$

defined by

$$\varphi(\theta) = (d\alpha - \alpha d)(\theta) \tag{2.14}$$

for any $\theta \in \mathfrak{g}^*$. Thus the curvature is the obstruction for α to be a homomorphism of graded differential algebras. It follows from the definitions that one has

$$i_X(\varphi(\theta)) = 0 \tag{2.15}$$

and

$$L_X(\varphi(\theta)) = \varphi(\theta \circ \mathrm{ad}(X)) \tag{2.16}$$

for any $X \in \mathfrak{g}$ and $\theta \in \mathfrak{g}^*$.

Notice that to give α as above is the same as to give an element A of $\mathfrak{g} \otimes \Omega^1$ such that

$$i_X(A) = X \tag{2.17}$$

and

$$L_X(A) = \mathrm{ad}(X)A \tag{2.18}$$

where here i_X is $I_\mathfrak{g} \otimes i_X$, L_X is $I_\mathfrak{g} \otimes L_X$ and $\mathrm{ad}(X)$ is $\mathrm{ad}(X) \otimes I_{\Omega^1}$. The mapping α being then $\theta \mapsto (\theta \otimes I_{\Omega^1})A$.

In the same veine, the curvature φ is the same as the element F of $\mathfrak{g} \otimes \Omega^2$ defined by

$$F = dA + \frac{1}{2}[A, A] \tag{2.19}$$

where $d = I_\mathfrak{g} \otimes d$ and $[\ ,\ ]$ is the graded commutator in $\mathfrak{g} \otimes \Omega$, the mapping φ being then $\theta \mapsto (\theta \otimes I_{\Omega^2})F$.

One has in view of (2.15) and (2.16)

$$i_X(F) = 0 \tag{2.20}$$

and

$$L_X(F) = \mathrm{ad}(X)F \tag{2.21}$$

for $X \in \mathfrak{g}$ with obvious notations.

In the standard example of Section 2.3 where Ω is the graded differential algebra of differential forms on a principal bundle $P(M, G)$ with structure group G such that $\mathrm{Lie}(G) = \mathfrak{g}$, an algebraic connection on Ω is an ordinary principal bundle connection on $P(M, G)$.

2.5 The Weil algebra of a Lie algebra

Let $\mathfrak{g}$ be a fixed Lie algebra and consider the operations of $\mathfrak{g}$ with connections in graded commutative differential algebras. There is a straightforward notion of morphism for such objects and one gets the category of $\mathfrak{g}$-*operations with connections*. A morphism is a homomorphism of graded differential algebras which intertwins the $\mathfrak{g}$-operations and which maps the connection on the connection.

There is a universal initial object $W(\mathfrak{g})$ in this category which is called the *Weil algebra of the Lie algebra* $\mathfrak{g}$. As graded commutative algebra $W(\mathfrak{g})$ is the tensor product $\wedge\mathfrak{g}^*\otimes S\mathfrak{g}^*$ of the exterior algebra $\wedge\mathfrak{g}^*$ of the dual vector space $\mathfrak{g}^*$ of $\mathfrak{g}$ with the symmetric algebra $S\mathfrak{g}^*$ where the degree $2n$ is given to the elements of $S^n\mathfrak{g}^*$.

Let θ be an element of $\mathfrak{g}^*$, we denote by $\alpha(\theta)$ the element $\theta\otimes\mathbb{1}$ of $W^1(\mathfrak{g})$ and by $\varphi(\theta)$ the element $\mathbb{1}\otimes\theta$ of $W^2(\mathfrak{g})$. It is clear that there is a unique differential on $W(\mathfrak{g})$ for which

$$d\alpha(\theta) = \alpha(d\theta) + \varphi(\theta)$$

for any $\theta\in\mathfrak{g}^*$. In fact $W(\mathfrak{g})$ can be defined as well by a change of generators as the free graded commutative differential algebra generated by $\alpha(\mathfrak{g}^*)$ in degree 1 and by $d\alpha(\mathfrak{g}^*)$ in degree 2 which is a contractible algebra [36].

One defines then the operation $X\mapsto i_X$ of $\mathfrak{g}$ in $W(\mathfrak{g})$ by defining i_X to be the unique antiderivation of $W(\mathfrak{g})$ such that

$$i_X(\alpha(\theta)) = \theta(X)$$

and

$$i_X(\varphi(\theta)) = 0$$

for any $\theta\in\mathfrak{g}^*$, $(X\in\mathfrak{g})$. One verifies that all the axioms for an operation of $\mathfrak{g}$ in the graded differential algebra $W(\mathfrak{g})$ are satisfied and that α is then a connection with curvature φ in $W(\mathfrak{g})$.

One passes to the notations A, F above by setting

$$A = (I_\mathfrak{g}\otimes\alpha)(\tilde{I}_g)\in\mathfrak{g}\otimes W^1(\mathfrak{g})$$

where $\tilde{I}_{\mathfrak{g}}$ is the identity mapping $I_{\mathfrak{g}}$ of $\mathfrak{g}$ onto itself considered as an element of $\mathfrak{g} \otimes \mathfrak{g}^*$. One has then

$$F = dA + \frac{1}{2}[A, A] = (I_{\mathfrak{g}} \otimes \varphi)(\tilde{I}_{\mathfrak{g}}) \in \mathfrak{g} \otimes W^2(\mathfrak{g})$$

for the curvature.

By the very definition of $W(\mathfrak{g})$, its cohomology is trivial and one can show by introducing the appropriate contracting homotopy that its invariant cohomology is also trivial. The basic differential subalgebra $W_B(\mathfrak{g})$ of $W(\mathfrak{g})$ is given by

$$W_B^{2n}(\mathfrak{g}) = \mathbb{1} \otimes \mathcal{I}^n(\mathfrak{g}) \text{ and } W_B^{2n+1}(\mathfrak{g}) = 0$$

where $\mathcal{I}^n(\mathfrak{g})$ denotes the vector space of all ad-invariant homogeneous polynomials of degree n on $\mathfrak{g}$. It follows that $W_B(\mathfrak{g})$ coincides with its cohomology, that is with the basic cohomology $H_B(W(\mathfrak{g}))$ of $W(\mathfrak{g})$.

The definition of $W(\mathfrak{g})$ implies that, given an operation of $\mathfrak{g}$ with connection in a graded commutative differential algebra Ω, there is a unique homomorphism of graded differential algebras of $W(\mathfrak{g})$ into Ω which is a morphism of $\mathfrak{g}$-operation with connection. It can be shown that this homomorphism induces in basic cohomology an homomorphism which does not depend on the connection of Ω but only depends on the operation of $\mathfrak{g}$ in Ω. This is the algebraic version of the Weil homomorphism. One recovers the familiar version by applying it to the standard example of Section 2.3, remembering that the basic cohomology of the Weil algebra $W(\mathfrak{g})$ is isomorphic to the algebra $\mathcal{I}(\mathfrak{g})$ of all ad-invariant polynomials on $\mathfrak{g}$. Thus if $P(M, G)$ is a principal bundle over M (the basis) with structure group G such that $\mathfrak{g} = \mathrm{Lie}(G)$, the Weil homomorphism is an algebra homomorphism from $\mathcal{I}(\mathfrak{g})$ into the de Rham cohomology $H(M)$ of (the basis) M such that the image of $\mathcal{I}^n(\mathfrak{g})$ is contained in $H^{2n}(M)$ for any $n \in \mathbb{N}$.

Let $P \in \mathcal{I}(\mathfrak{g})$ be an ad-invariant polynomial on $\mathfrak{g}$. Then $\mathbb{1} \otimes P \in W(\mathfrak{g})$ is closed and invariant (in fact basic) so in view of the triviality of the invariant cohomology of $W(\mathfrak{g})$, one has $\mathbb{1} \otimes P = dQ$ with $Q \in W_I(\mathfrak{g})$. Let

$$\rho : W(\mathfrak{g}) \to \wedge \mathfrak{g}^*$$

be the canonical projection. The image $\rho(Q)$ is an invariant form in $\wedge \mathfrak{g}^*$ and it is not hard to show that it is independent of the choice of Q as above. The

corresponding linear mapping

$$\gamma : \mathcal{I}(\mathfrak{g}) \to \wedge_I \mathfrak{g}^*$$

from $\mathcal{I}(\mathfrak{g})$ into the space of invariant forms on $\mathfrak{g}$ is *the Cartan map*. One has

$$\gamma(\mathcal{I}^n(\mathfrak{g})) \subset \wedge_I^{2n-1} \mathfrak{g}^*$$

for any $n \geqslant 1$.

3 Extension from $\mathfrak{g}$ to $U(\mathfrak{g})$

This section is devoted to the extension of an operation of a Lie algebra $\mathfrak{g}$ in a graded differential algebra Ω from the Lie algebra $\mathfrak{g}$ to its universal enveloping algebra $U(\mathfrak{g})$. Needless to say our first goal in the sequel will be to state the axioms defining operations of Hopf algebras in graded differential algebras.

3.1 From L_X for $X \in \mathfrak{g}$ to L_h for $h \in U(\mathfrak{g})$

We first extend the L_X ($X \in \mathfrak{g}$). The linear mapping $X \mapsto L_X$ is, in view of (2.5), a representation of the Lie algebra $\mathfrak{g}$ in Ω. It follows from the universal defining property of $U(\mathfrak{g})$ that L extends uniquely as a representation $h \mapsto L_h$ of the unital associative algebra $U(\mathfrak{g})$ in Ω, i.e. here as a homomorphism of $U(\mathfrak{g})$ into the algebra $\mathrm{End}^0(\Omega)$ of endomorphisms of degree 0 of Ω. This extension will be refered to as *the canonical extension of L to $U(\mathfrak{g})$*.

Let us recall that $U(\mathfrak{g})$ is not only an algebra but that it is a Hopf algebra with unique coproduct Δ, counit ε and antipode S such that

$$\Delta X = X \otimes \mathbb{1} + \mathbb{1} \otimes X, \quad \varepsilon(X) = 0, \quad S(X) = -X \tag{3.1}$$

for $X \in \mathfrak{g}$.

Proposition 1. *The canonical extension of L to $U(\mathfrak{g})$ has the following properties :*
(a) $L_h d = dL_h$ *for any* $h \in U(\mathfrak{g})$
(b) $L_h(\mathbb{1}) = \varepsilon(h)\mathbb{1}$ *for any* $h \in U(\mathfrak{g})$ *where* $\mathbb{1}$ *is the unit of* Ω,
(c) $L_h(\alpha\beta) = \sum_i L_{h_i^{(1)}}(\alpha) L_{h_i^{(2)}}(\beta)$ *for any* $h \in U(\mathfrak{g})$ *with* $\Delta h = \sum_i h_i^{(1)} \otimes h_i^{(2)}$ *and for any* $\alpha, \beta \in \Omega$.

Proof (a) is clear since $L_{\mathbb{1}}$ is the identity mapping of Ω onto itself, the L_X commute with d for $X \in \mathfrak{g}$ and since the unit $\mathbb{1} \in U(\mathfrak{g})$ and the $X \in \mathfrak{g}$ generate $U(\mathfrak{g})$.

(b) follows from the fact that $L_X(\mathbb{1}) = 0$ for $X \in \mathfrak{g}$ so $L_h(\mathbb{1}) = 0$ for $h \in \mathrm{Ker}(\varepsilon)$ and from $L_{\mathbb{1}}(\mathbb{1}) = \mathbb{1}$ (since $L_{\mathbb{1}}$ is the identity mapping of Ω).

(c) follows from the fact that it holds for $h = \mathbb{1}(\in U(\mathfrak{g}))$ and for $h = X \in \mathfrak{g}$ and that furthermore both $h \mapsto L_h$ and $h \mapsto \Delta h$ are multiplicative homomorphisms. □

Let us summarize the properties of the mapping $h \mapsto L_h$.

$$L_h d = dL_h, \quad \forall h \in U(\mathfrak{g}) \tag{3.2}$$

$$L_{h_1} L_{h_2} = L_{h_1 h_2}, \quad \forall h_1, h_2 \in U(\mathfrak{g}) \tag{3.3}$$

$$L_{\mathbb{1}} = I_\Omega \tag{3.4}$$

$$L_h \circ m = m \circ (L \otimes L)_{\Delta h}, \quad \forall h \in U(\mathfrak{g}) \tag{3.5}$$

$$L_h(\mathbb{1}) = \varepsilon(h)\mathbb{1}, \quad \forall h \in U(\mathfrak{g}) \tag{3.6}$$

where I_Ω is the identity mapping of Ω onto itself, m is the product of Ω and the $\mathbb{1}$ are either the unit of $U(\mathfrak{g})$ or the one of Ω.
One also has the following identity which extends (2.3)

$$i_X L_{Y_1 \dots Y_n} = \sum_p \sum_{(i,j) \in (p,n-p) \text{ shuffles}} L_{Y_{i_1} \dots Y_{i_p}} i_{[\dots[X, Y_{j_1}], \dots, Y_{j_{n-p}}]} \tag{3.7}$$

for X and the Y_k in $\mathfrak{g}$. By using the fact that one has

$$\Delta(Y_1, \dots Y_n) = \sum_p \sum_{(i,j) \in (p,n-p) \text{shuffles}} Y_{i_1} \dots Y_{i_p} \otimes Y_{j_1} \dots Y_{j_{n-p}} \tag{3.8}$$

one sees that (3.7) implies the more general identity

$$i_X L_h = \sum_i L_{h_i^{(1)}} i_{\mathrm{ad}(h_i^{(2)})X} \tag{3.9}$$

for any $h \in U(\mathfrak{g})$ with $\Delta h = \sum_i h_i^{(1)} \otimes h_i^{(2)}$ and any $X \in \mathfrak{g}$. In the above, the right adjoint action ad is defined by

$$\mathrm{ad}(h)g = \sum_i S(h_i^{(1)})g h_i^{(2)} \tag{3.10}$$

for any $g, h \in U(\mathfrak{g})$ with $\Delta h = \sum_i h_i^{(1)} \otimes h_i^{(2)}$ and one has

$$\mathrm{ad}(h)\mathfrak{g} \subset \mathfrak{g} \tag{3.11}$$

for any $h \in U(\mathfrak{g})$, (stability of primitive elements).

3.2 Multiplicative constraints for the extension of i

In the sequel of this section, we wish to extend $X \mapsto i_X$ as a linear mapping $h \mapsto i_h$ of $U(\mathfrak{g})$ into $\mathrm{End}(\Omega)$. In this subsection we discuss the consequences of the following natural constraints required on the i_h

$$i_{X_1\dots X_{k-1}X_kX_{k+1}X_{k+2}\dots X_n} - i_{X_1\dots X_{k-1}X_{k+1}X_kX_{k+2}\dots X_n} = i_{X_1\dots X_{k-1}[X_k,X_{k+1}]X_{k+2}\dots X_n} \tag{3.12}$$

for $X_i \in \mathfrak{g}, 0 \leqslant k \leqslant n-1$ and $n \geqslant 2$. These constraints will be refered to as the *multiplicative constraints* for i. Since the i_X for $X \in \mathfrak{g}$ are of degree -1, the consistency with (3.12) requires that the i_h are also of degree -1 for any $h \in U(\mathfrak{g})$. Thus we impose that

$$i_h \in \mathrm{End}^{-1}(\Omega) \tag{3.13}$$

for any $h \in U(\mathfrak{g})$ and for $i_{\mathbb{1}}$ (i.e. for $h = \mathbb{1}$) we impose that

$$i_{\mathbb{1}} = 0 \tag{3.14}$$

which is the only natural possibility.

Let us try to solve (3.12) by induction on n starting from the original i_X for $X \in \mathfrak{g}$. Assume that we know the n-linear mappings $(X_1, \dots, X_n) \mapsto i_{X_i\dots X_n}$ of $\mathfrak{g}$ into $\mathrm{End}^{-1}(\Omega)$ for $1 \leqslant n < N$, the arbitrariness of $(X_1, \dots, X_N) \mapsto i_{X_1\dots X_N}$ satisfying (3.12) is the addition of a completely symmetric N-linear mapping $(X_1, \dots, X_N) \mapsto S^{(N)}_{X_1,\dots,X_N}$ of $\mathfrak{g}$ into $\mathrm{End}^{-1}(\Omega)$, i.e. $S^{(N)} \in \mathrm{End}^{-1}(\Omega) \otimes S^N\mathfrak{g}^*$.
Thus the arbitrariness of the solution of the multiplicative constraints (3.12)

starting with i_X for $X \in \mathfrak{g}$ is a sequence $(S^{(n)})_{n\geqslant 2}$ of symmetric multilinear (n-linear for $S^{(n)}$) mappings of $\mathfrak{g}$ into $\mathrm{End}^{-1}(\Omega)$.
On the other hand, the existence of a solution of (3.12) is straightforward by chosing an ordered basis (e_μ) of $\mathfrak{g}$ since then one can prescribe arbitrarily the $i_{e_{\mu_1}\dots e_{\mu_n}}$ for $\mu_1 \geqslant \mu_2 \geqslant \dots \geqslant \mu_n$ and $n \geqslant 2$, the permuted $i_{e_{\mu_{\sigma(1)}}\dots e_{\mu_{\sigma(n)}}}$ being then inductively defined by (3.12).

3.3 Differential constraints

By definition one has $L_X = di_X + i_X d$ for $X \in \mathfrak{g}$. For $n = 2$ equation (3.12) reads

$$i_{X_1X_2} - i_{X_2X_1} = i_{[X_1,X_2]}, \quad X_1, X_2 \in \mathfrak{g}$$

so that by taking the anticommutator with d on both sides one obtains

$$\begin{aligned}(di_{X_1X_2} + i_{X_1X_2}d) - (di_{X_2X_1} + i_{X_2X_1}d) &= di_{[X_1,X_2]} + i_{[X_1,X_2]}d \\ &= L_{[X_1,X_2]} = L_{X_1X_2} - L_{X_2X_1}\end{aligned}$$

which implies

$$L_{X_1X_2} = di_{X_1X_2} + i_{X_1X_2}d + S'_{X_1,X_2}$$

with $S'_{X_1,X_2} = S'_{X_2,X_2}$ and $dS'_{X_1,X_2} = S'_{X_1,X_2}d$.
If $S'_{X_1,X_2} = dS_{X_1,X_2} + S_{X_1,X_2}d$ with $S_{X_1X_2} = S_{X_2X_1}$ one can absorb it by a redefinition of $i_{X_1X_2}$ as in §3.2. On the other hand if the homology of Ω is trivial in degre zero then one can take $S'_{X_1X_2}$ of this form. All this suggests that one can impose $L_{X_1X_2} = di_{X_1X_2} + i_{X_1X_2}d$. More generally if $L_{X_1\dots X_{n-1}} = di_{X_1\dots X_{n-1}} + i_{X_1\dots X_{n-1}}d$ for $X_i \in \mathfrak{g}$, then by taking the anticommutator with d on both sides of (3.12) one obtains

$$L_{X_1\dots X_n} = di_{X_1\dots X_n} + i_{X_1\dots X_n}d + S'_{X_1\dots,X_n}$$

where $S'_{X_1,\dots,X_n}$ is completely symmetric in the X_k and commutes with d. If $S'_{X_1,\dots,X_n} = dS_{X_1,\dots,X_n} + S_{X_1,\dots,X_n}d$, with $S_{X_1\dots X_n}$ completely symmetric in the X_k, then it can be absorbed by a redefinition of $i_{X_1\dots X_n}$, etc. so that it is natural to impose

$$L_{X_1\dots X_n} = di_{X_i\dots X_n} + i_{X_1\dots X_n}d$$

for $X_i \in \mathfrak{g}$ and $n \geqslant 1$, which means

$$L_h = di_h + i_h d, \quad \forall h \in \mathrm{Ker}(\varepsilon)$$

and finally

$$L_h = di_h + i_h d + \varepsilon(h) I_\Omega \tag{3.15}$$

for any $h \in U(\mathfrak{g})$ where I_Ω is the identity mapping of Ω onto itself. In the following to simplify the notations, we shall write ε_h to denote the multiplication by $\varepsilon(h) \in \mathbb{K}$ when no confusion arises. So here we write

$$L_h = di_h + i_h d + \varepsilon_h \tag{3.16}$$

for (3.15), $\forall h \in U(\mathfrak{A})$. These conditions (3.16) will be refered to as the *differential constraints* for i.

3.4 Extension of the Cartan relation

Equation (3.9) which extends the Cartan relation (2.3) suggest to impose more generally

$$i_g L_h = \sum_i L_{h_i^{(1)}} i_{\mathrm{ad}(h_i^{(2)})g} \tag{3.17}$$

for any $g, h \in U(\mathfrak{g})$ with $\Delta h = \sum_i h_i^{(1)} \otimes h_i^{(2)}$ where ad denotes the right adjoint action defined by Equation (3.10). By using the general axiom for the antipode and the coassociativity of the coproduct, one shows easily that Equation (3.17) is equivalent to

$$\sum_i L_{S(h_i^{(1)})} i_g L_{h_i^{(2)}} = i_{\mathrm{ad}(h)g} \tag{3.18}$$

for any $g, h \in U(\mathfrak{g})$ with $\Delta h = \sum_i h_i^{(1)} \otimes h_i^{(2)}$. This last equation can be interpreted as (right) ad equivariance of $g \mapsto i_g$.

Notice, as a further partial justification of (3.17), that Equation (3.16) together with Equation (3.17) imply

$$L_g L_h = \sum_i L_{h_i^{(1)}} L_{\mathrm{ad}(h_i^{(2)})g} \tag{3.19}$$

and that this relation (3.19) is also implied by (3.3), that is by $L_g L_h = L_{gh}$ for any $g, h \in U(\mathfrak{g})$, together with the usual axiom for the antipode S.

The following remarks are in order.

Remarks

1. The Cartan relation (2.3) extends as well as

$$L_h i_X = \sum_i i_{\mathrm{ad'}(h_i^{(1)})X} L_{h_i^{(2)}}$$

for $X \in \mathfrak{g}$ and $h \in U(\mathfrak{g})$ with $\Delta h = \sum_i h_i^{(1)} \otimes h_i^{(2)}$ where the left adjoint action ad' is defined by

$$\mathrm{ad'}(h)g = \sum_i h_i^{(1)} g S(h_i^{(2)}) \tag{3.20}$$

for $g \in U(\mathfrak{g})$. So we could as well impose

$$L_h i_g = \sum_i i_{\mathrm{ad'}(h_i^{(1)})g} L_{h_i^{(2)}} \tag{3.21}$$

which implies

$$L_h L_g = \sum_i L_{\mathrm{ad'}(h_i^{(1)})g} L_{h_i^{(2)}} \tag{3.22}$$

instead of (3.19). Furthermore (3.22) is also implied by (3.3).

2. In the case of the Hopf algebra $U(\mathfrak{g})$, one has $S^2 = I$ and $U(\mathfrak{g})$ is generated by $\mathfrak{g}$ so in this case one can expect that (3.19) and (3.22) are both equivalent to (3.3) but this is certainly not the case for an arbitrary Hopf algebra $\mathcal{H}$.

With these remarks in mind, we adopt (3.17) and (3.16) for the extension to $U(\mathfrak{g})$ of the operation of $\mathfrak{g}$ in Ω.

3.5 Extension of the antiderivation property of i_X

It remains to generalize from $X \in \mathfrak{g}$ to $g \in U(\mathfrak{g})$ the antiderivation property of i_X. For this we impose

$$i_g(\alpha\beta) = \sum_i i_{g_i^{(1)}}(\alpha) L_{g_i^{(2)}}(\beta) + (-1)^a \alpha i_g(\beta) \tag{3.23}$$

for $g \in U(\mathfrak{g})$ with $\Delta g = \sum_i g_i^{(1)} \otimes g_i^{(2)}$, $\alpha \in \Omega^a$ and $\beta \in \Omega$. This can be written as

$$i_g \circ m = m \circ (i \otimes L + \varepsilon \otimes i)_{\Delta g} \tag{3.24}$$

where m is the product of Ω, (remembering that ε_h is the multiplication by $\varepsilon(h)$ in Ω for $h \in U(\mathfrak{g})$).

To justify this choice, we notice the following :

1. By using it with (3.16) one obtains $L_g(\alpha\beta) = \sum_i L_{g_i^{(1)}}(\alpha)L_{g_i^{(2)}}(\beta)$ which is Property (c) of Proposition 1.

2. One has $i_g((\alpha\beta)\gamma) = i_g(\alpha(\beta\gamma))$ for any $\alpha, \beta, \gamma \in \Omega$, (i.e. compatibility with the associativity of the product of Ω).

3. For $X \in \mathfrak{g}$, (3.23) reduces to the antiderivation property of i_X.

It is worth noticing here that there is another choice satisfying Properties 1, 2, 3 above which reads

$$i_g(\alpha\beta) = i_g(\alpha)\beta + \sum_i (-1)^a L_{g_i^{(1)}}(\alpha) i_{g_i^{(2)}}(\beta) \tag{3.25}$$

and that, under (3.16), these are essentially the unique choices.

The choice of (3.23) together with (3.17) is coherent and dictated by the fact that the classical situation of a principal bundle corresponds to a right action.

4 Operations of Hopf algebras

We are now ready to define, more generally, operations of Hopf algebras in graded differential algebras.

4.1 Definition

Let $\mathcal{H}$ be a Hopf algebra with coproduct Δ, counit ε and antipode S and let Ω be a graded differential algebra with differential d.
An operation of the Hopf algebra $\mathcal{H}$ in the graded differential algebra Ω is a linear mapping $h \mapsto i_h$ of $\mathcal{H}$ into the vector space $\mathrm{End}^{-1}(\Omega)$ of homogeneous linear endomorphisms of degree -1 of Ω satisfying

$$i_{\mathbb{1}} = 0 \tag{4.1}$$

and such that by setting for $h \in \mathcal{H}$

$$L_h = di_h + i_h d + \varepsilon(h) I_\Omega \tag{4.2}$$

where I_Ω is the identity mapping of Ω, one has for any $h \in \mathcal{H}$ with $\Delta h = \sum_i h_i^{(1)} \otimes h_i^{(2)}$

$$i_h(\alpha\beta) = \sum_i i_{h_i^{(1)}}(\alpha) L_{h_i^{(2)}}(\beta) + (-1)^a \alpha i_h(\beta) \tag{4.3}$$

for $\alpha \in \Omega^a, \beta \in \Omega$ and

$$i_g L_h = \sum_i L_{h_i^{(1)}} i_{\mathrm{ad}(h_i^{(2)})g} \tag{4.4}$$

for $g \in \mathcal{H}$, where the right adjoint action ad is defined (as before) by

$$\mathrm{ad}(h)g = \sum_i S(h_i^{(1)}) g h_i^{(2)} \tag{4.5}$$

for $g, h \in \mathcal{H}$ (with Δh as above) and

$$L_h L_g = L_{hg} \tag{4.6}$$

for $h, g \in \mathcal{H}$.

By using the associativity of Δ the definitions of the counit ε and of the antipode S, one verifies that Equation (4.4) is equivalent to

$$\sum_i L_{S(h_i^{(1)})} i_g L_{h_i^{(2)}} = i_{\mathrm{ad}(h)g} \tag{4.7}$$

which implies by using (4.2) that

$$\sum_i L_{S(h_i^{(1)})} L_g L_{h_i^{(2)}} = L_{\mathrm{ad}(h)g} \tag{4.8}$$

for any $g, h \in \mathcal{H}$ with $\Delta h = \sum_i h_i^{(1)} \otimes h_i^{(2)}$.
One sees that Equation (4.8) is also implied by the axiom (4.6).
Equations (4.7) and (4.8) mean the ad-equivariance of the mappings $h \mapsto i_h$ and $h \mapsto L_h$.

From (4.2) and (4.1) it follows that one has

$$L_{\mathbb{1}} = I_\Omega \tag{4.9}$$

and

$$L_h(\mathbb{1}) = \varepsilon(h)\mathbb{1} \tag{4.10}$$

where in (4.9) $\mathbb{1}$ is the unit of $\mathcal{H}$ while in (4.10) $\mathbb{1}$ is the unit of Ω.

From (4.2) and (4.3) it follows that one has

$$L_h(\alpha\beta) = \sum_i L_{h_i^{(1)}}(\alpha) L_{h_i^{(2)}}(\beta) \tag{4.11}$$

for $h \in \mathcal{H}$ with $\Delta h = \sum_i h_i^{(1)} \otimes h_i^{(2)}$ and $\alpha, \beta \in \Omega$.

Finally

$$L_h d = d L_h \tag{4.12}$$

follows from (4.2).

We shall set in the following as in the last section

$$\varepsilon_h = \varepsilon(h) I_\Omega \tag{4.13}$$

for $h \in \mathcal{H}$ in order to simplify the notations.

4.2 Invariance, horizontality and basicity

Given an operation of $\mathcal{H}$ in Ω as in 4.1, an element α of Ω will be said to be *invariant* if one has

$$L_h(\alpha) = \varepsilon(h)\alpha \tag{4.14}$$

for any $h \in \mathcal{H}$, α will be said to be *horizontal* if one has

$$i_h(\alpha) = 0 \tag{4.15}$$

for any $h \in \mathcal{H}$ and α will be said to be *basic* if it is both invariant and horizontal that is if one has

$$\begin{cases} L_h(\alpha) & = & \varepsilon(h)\alpha \\ i_h(\alpha) & = & 0 \end{cases} \tag{4.16}$$

for any $h \in \mathcal{H}$.

It follows from (4.2) and (4.11) that the set Ω_I of all invariant elements of Ω is a graded differential subalgebra of Ω, it follows from (4.3) and (4.4) that the set Ω_H of all horizontal elements of Ω is a graded subalgebra of Ω which is stable by the L_h for $h \in \mathcal{H}$ and it follows from (4.2) again that the set $\Omega_B = \Omega_I \cap \Omega_H$ of all basic elements is a graded differential subalgebra of Ω_I and therefore also of Ω. The cohomology $H_I(\Omega)$ of Ω_I will be refered to as the invariant cohomology of Ω while the cohomology $H_B(\Omega)$ of Ω_B will be refered to as the basic cohomology of Ω for the operation of $\mathcal{H}$ in Ω.

4.3 Operations of the Lie algebra $\mathfrak{g}$ and of the Hopf algebra $U(\mathfrak{g})$

Let $\mathfrak{g}$ be a Lie algebra and let $h \mapsto i_h$ be an operation of the Hopf algebra $U(\mathfrak{g})$ in the graded differential algebra Ω. It is clear that by restriction to $\mathfrak{g} \subset U(\mathfrak{g})$ one obtains an operation $X \mapsto i_X$ of the Lie algebra $\mathfrak{g}$ in the graded differential algebra Ω, (in the sense of §2.1). For the converse, one has the following result.

Proposition 2. *Let Ω be a graded differential algebra. Assume that Ω is generated in degree 0 as graded differential algebra, that is that the smallest graded differential subalgebra of Ω which contains Ω^0 is Ω itself. Then an operation of the Lie algebra $\mathfrak{g}$ in Ω has a unique extension as an operation of the Hopf algebra $U(\mathfrak{g})$ in $\mathfrak{g}$.*

Proof. Any element of Ω is a linear combination of terms of the form

$$x_0 dx_1 \dots dx_n$$

with $x_\alpha \in \Omega^0$. Let $h \mapsto i_h$ be an operation of $U(\mathfrak{g})$ in Ω. One has

$$i_h(x_0 dx_1 \dots dx_n) = x_0 i_h(dx_1 \dots dx_n)$$

and $i_h(dx_\alpha) = L_h(x_\alpha) - \varepsilon(h) x_\alpha$. It follows that the operation is completely specified by the L_h in view of (4.3). On the other hand L_h is completely specified by the L_X for $X \in \mathfrak{g}$ in view of (4.6) since $\mathfrak{g}$ generates $U(\mathfrak{g})$. Thus, starting from the i_X with $X \in \mathfrak{g}$ one constructs the L_h for $h \in U(\mathfrak{g})$ and the i_h for $h \in U(\mathfrak{g})$ by using the above formulae. □

Remarks

1. Under the conditions of Proposition 2, the notions of invariance, horizontality and basicity are the same for the operation of the Lie algebra $\mathfrak{g}$ and for the operation of the Hopf algebra $U(\mathfrak{g})$.

2. If Ω is the algebra of smooth differential forms on a smooth manifold, it satisfies the condition of Proposition 2. This is also the case if Ω is a quotient of the universal differential calculus over an associative algebra, [10], [27], (see also in [21]).
   This is however not the case for $\Omega = \wedge\mathfrak{g}^*$ or, more generally for the graded differential algebras associated by Koszul duality to the Lie prealgebras, [23]. Indeed these later graded differential algebras are generated in degree 1.

4.4 The category of $\mathcal{H}$-operations

In this subsection, $\mathcal{H}$ is a fixed Hopf algebra and we introduce a notion of morphism for operations of $\mathcal{H}$ in graded differential algebras. Operations of $\mathcal{H}$ in graded differential algebras will be also refered to as *$\mathcal{H}$-operations*.

Given an operation i of $\mathcal{H}$ in Ω and an operation i' of $\mathcal{H}$ in Ω', a *morphism of $\mathcal{H}$-operation* from (i,Ω) to (i',Ω') is a homomorphism $f:\Omega\rightarrow\Omega'$ of graded differential algebra which satisfies

$$f(i_h(\omega)) = i'_h(f(\omega))$$

for any $h\in\mathcal{H}$ and $\omega\in\Omega$.

The $\mathcal{H}$-operations and their morphisms form a category which will be refered to as *the category of $\mathcal{H}$-operations*.

4.5 The Hopf superalgebra formulation

Let $\mathcal{H}$ be a Hopf algebra. Then a *$\mathcal{H}$-algebra* is an algebra $\mathcal{A}$ endowed with an action $\mathcal{H}\otimes\mathcal{A}\rightarrow\mathcal{A}$ of $\mathcal{H}$, $h\otimes a\mapsto ha$ satisfying

$$h(ga) = (hg)a, \quad \forall h,g\in\mathcal{H},\ a\in\mathcal{A} \tag{4.17}$$

and

$$h(ab) = \sum_i (h_i^{(1)}a)(h_i^{(2)}b), \quad \forall h\in\mathcal{H},\ a,b\in\mathcal{A} \tag{4.18}$$

with $\Delta h = \sum_i h_i^{(1)} \otimes h_i^{(2)}$; [18], [15].

There is an obvious graded (super) version of the above notion : Let $\mathcal{H}$ be a $\mathbb{Z}$-graded (super) Hopf algebra, then a *graded $\mathcal{H}$-algebra* is a $\mathbb{Z}$-graded algebra $\mathcal{A}$ with a *homogeneous* action $\mathcal{H} \otimes \mathcal{A} \to \mathcal{A}$ satisfying (4.17) and (4.18). Notice that the term "super" is not necessary since we use here the Koszul sign convention.

Let $\mathcal{H}$ be an ordinary (non graded) Hopf algebra. To $\mathcal{H}$ one associates a $\mathbb{Z}$-graded Hopf algebra $\widehat{\mathcal{H}}$ in the following manner. In degree 1, $\widehat{\mathcal{H}}_1$ is 1-dimensional generated by an element δ with

$$\delta^2 = 0, \quad \Delta\delta = \delta \otimes \mathbb{1} + \mathbb{1} \otimes \delta, \quad \varepsilon(\delta) = 0, \quad S(\delta) = -\delta$$

where Δ is the coproduct, ε is the counit and S is the antipode of $\widehat{\mathcal{H}}$. In degree 0, $\widehat{\mathcal{H}}_0$ is isomorphic to $\mathcal{H}$ as Hopf algebra and we denote by

$$\Lambda : \mathcal{H} \mapsto \widehat{\mathcal{H}}_0, \quad h \mapsto \Lambda_h$$

the corresponding isomorphism. The relations with δ are

$$[\delta, \Lambda_h] = 0$$

for $h \in \mathcal{H}$. In degree -1, $\widehat{\mathcal{H}}_{-1}$ is isomorphic as vector space to the quotient $\mathcal{H}/\mathbb{K}\mathbb{1}$ and we denote by

$$y : \mathcal{H} \mapsto \widehat{\mathcal{H}}_{-1}, \quad h \mapsto y_h$$

the corresponding linear mapping which vanishes on $\mathbb{1} \in \mathcal{H}$ (i.e. $y_{\mathbb{1}} = 0$). The relations, the coproduct and the antipode of y_h are given by

$$\begin{array}{lcl} y_g \Lambda_h & = & \sum_i \Lambda_{h_i^{(1)}} y_{\mathrm{ad}(h_i^{(2)})g} \\ [\delta, y_h] & = & \Lambda_h - \varepsilon(h)\mathbb{1} \\ \Delta y_h & = & \sum_i y_{h_i^{(1)}} \otimes \Lambda_{h_i^{(2)}} + \mathbb{1} \otimes y_h \\ S(y_h) & = & -\sum_i y_{h_i^{(1)}} \Lambda_{S(h_i^{(2)})} \end{array}$$

for $g, h \in \mathcal{H}$ with $\Delta h = \sum_i h_i^{(1)} \otimes h_i^{(2)}$ while one has $\varepsilon(y_h) = 0$. The graded Hopf algebra $\widehat{\mathcal{H}}$ is generated by $\widehat{\mathcal{H}}_1 \oplus \widehat{\mathcal{H}}_0 \oplus \widehat{\mathcal{H}}_{-1}$ and there are no other relations in $\widehat{\mathcal{H}}$.

Let us consider an operation of $\mathcal{H}$ in a graded differenial algebra (Ω, d). Then $\delta \mapsto d, \Lambda_h \mapsto L_h$ and $y_h \mapsto i_h$ define a structure of graded $\widehat{\mathcal{H}}$-algebra on Ω. It is clear that this correspondence allows to identify the notion of $\mathcal{H}$-operation with the notion of positively graded $\widehat{\mathcal{H}}$-algebra.

This is the counterpart in our noncommutative context of the classical graded Lie superalgebra formulation of the operations of Lie algebras [34], [26], [1].

To conclude this Section 4, we notice that the analogues of the classical standard example of §2.3 are the appropriate differential calculi over the noncommutative principal bundles. Among these noncommutative principal bundles, let us mention the principal bundles in [32] on the noncommutative manifolds of [13], [12]: in particular the $SU(2)$ principal bundle [31] over a noncommutative 4-sphere. The $SU_q(2)$-principal bundle over a quantum 4-sphere in [4] uses in a crucial way the covariant calculi of [37], [38] and generalizes the $U(1)$-fibration [5], [6] over the quantum 2-sphere. It is also worth noticing here that the theory of Hopf-Galois extensions includes a general formulation of noncommutative principal bundles, see e.g. [35] and references therein.

5 Theory of connections

In this section we introduce and study a noncommutative generalization of the theory of algebraic connections on the operations of Lie algebra [8], [25].

5.1 The graded differential algebra $C(\mathcal{H})$

Although in the sequel $\mathcal{H}$ is a Hopf algebra, in this subsection only its algebra structure is involved. At the end of this subsection the augmentation ε (counits) of $\mathcal{H}$ will also play a role.

Let $C(\mathcal{H}) = \oplus_n C^n(\mathcal{H})$ be the graded algebra of multilinear forms on $\mathcal{H}$, that is one has

$$C^n(\mathcal{H}) = (\mathcal{H}^{\otimes^n})^*$$

and the product is the tensor product of multilinear forms.

The product $\mu : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ of $\mathcal{H}$ induces by transposition the linear mapping

$$\mu^t : \mathcal{H}^* \to (\mathcal{H} \otimes \mathcal{H})^*$$

so $-\mu^t$ (the minus sign is here to match the usual convention) is a linear mapping of $C^1(\mathcal{H})$ into $C^2(\mathcal{H})$ which has an extension

$$d_0 : C(\mathcal{H}) \to C(\mathcal{H})$$

as an antiderivation of degree 1 of $C(\mathcal{H})$ given by

$$d_0(\Psi)(h_0, h_1, \dots, h_n) = \sum_{1 \leqslant k \leqslant n} (-1)^k \Psi(h_0, \dots, h_{k-2}, h_{k-1}h_k, h_{k+1}, \dots, h_n) \tag{5.1}$$

for $\Psi \in C^n(\mathcal{H})$ and $h_0, \dots, h_n \in \mathcal{H}$. The associativity of the product of $\mathcal{H}$ is equivalent to

$$d_0^2 = 0 \tag{5.2}$$

so that $(C(\mathcal{H}), d_0)$ is a graded differential algebra.

Proposition 3. *The cohomology $H(C(\mathcal{H}), d_0)$ of $(C(\mathcal{H}), d_0)$ is trivial, that is one has*

$$H^0(C(\mathcal{H}), d_0) = \mathbb{K} \quad \text{and} \quad H^n(C(\mathcal{H}), d_0) = 0$$

for $n \geqslant 1$.

Proof. $H^0(C(\mathcal{H}), d_0) = \mathbb{K}$ is obvious since $C^0(\mathcal{H}) = \mathbb{K}\mathbb{1}$. If $\Psi \in C^n(\mathcal{H})$ with $n \geqslant 1$, one has by setting for $h_i \in \mathcal{H}$

$$K(\Psi)(h_1, \dots, h_{n-1}) = \Psi(\mathbb{1}, h_1, \dots, h_{n-1})$$

the identity

$$(dK + Kd)(\Psi) = \Psi$$

which implies $H^n(C(\mathcal{H}), d_0) = 0$ for $n \geqslant 1$. $\square$

The construction of the graded differential algebra $(C(\mathcal{H}), d_0)$ works as well for any unital associative algebra and the above result is valid (and classical) in this context (it is dual to the acyclic bar complex, see e.g.[33]).

The counit ε gives a structure of $\mathcal{H}$-bimodule to the ground field $\mathbb{K}$, this $\mathcal{H}$-bimodule will be refered to as *the trivial bimodule* $\mathbb{K}$. Thus $C(\mathcal{H})$ is also the space of Hochschild cochains of $\mathcal{H}$ with coefficients in the trivial bimodule $\mathbb{K}$. The corresponding Hochschild differential d reads

$$d\omega = d_0\omega + \varepsilon\omega + (-1)^{n+1}\omega\varepsilon \tag{5.3}$$

for $\omega \in C^n(\mathcal{H})$.

In contrast to the cohomology of $C(\mathcal{H})$ for d_0 (see Proposition 3), the cohomology of d is nontrivial in general. For instance in the case $\mathcal{H} = U(\mathfrak{g})$, it coincides with the cohomology of the Lie algebra $\mathfrak{g}$. Thus, from this point of view $C(\mathcal{H})$ equipped with the differential d is the analogue of $\wedge\mathfrak{g}^*$ equipped with the Koszul differential. In the following $C(\mathcal{H})$ endowed with the differential d will be refered to as *the graded differential algebra* $C(\mathcal{H})$.

5.2 The operation of $\mathcal{H}$ in $C(\mathcal{H})$

One defines an operation of the Hopf algebra $\mathcal{H}$ in the graded differential algebra $C(\mathcal{H})$ in the following manner. Let $\Psi \in C^n(\mathcal{H})$ be a n-linear form on $\mathcal{H}$ and let us define $i_h(\Psi) \in C^{n-1}(\mathcal{H})$ for $h \in \mathcal{H}$ by

$$\begin{aligned} & i_h(\Psi)(g_1, \dots, g_{n-1}) = \\ & \sum_{p=0}^{n-2} (-1)^p \sum_{i_p} \Psi(g_1, \dots, g_p, h^{(1)}_{i_p} - \varepsilon(h^{(1)}_{i_p})\mathbb{1}, \mathrm{ad}(h^{(2)}_{i_p}) g_{p+1}, \dots, \mathrm{ad}(h^{(n-p)}_{i_p}) g_{n-1}) \\ & + (-1)^{n-1}\Psi(g_1, \dots, g_{n-1}, h - \varepsilon(h)\mathbb{1}) \end{aligned} \tag{5.4}$$

for $g_k \in \mathcal{H}$, where we have set

$$(\Delta \otimes I_{\mathcal{H}}^{\otimes^{n-p-2}}) \dots (\Delta \otimes I_{\mathcal{H}})\Delta h = \sum_{i_p} h^{(1)}_{i_p} \otimes h^{(2)}_{i_p} \otimes \dots \otimes h^{(n-p)}_{i_p} \tag{5.5}$$

for the iterated coproducts (which occur of course only for $n \geqslant 2$).

One verifies that this defines a $\mathcal{H}$-operation and that $L_h = di_h + i_h d + \varepsilon_h$ is given by

$$L_h(\Psi)(g_1,\dots,g_n)=\sum_{i_0}\Psi(\mathrm{ad}(h^{(1)}_{i_0})g_1,\dots,\mathrm{ad}(h^{(n)}_{i_0})g_n) \tag{5.6}$$

with obvious notations.

One verifies that one has

$$i_h d + d i_h = i_h d_0 + d_0 i_h \tag{5.7}$$

which implies that $h\mapsto i_h$ is also an operation of $\mathcal{H}$ in the graded differential algebra $(C(\mathcal{H}),d_0)$. Concerning the invariant cohomology of $(C(\mathcal{H}),d_0)$ one has the following result.

Proposition 4. *The invariant cohomology $H_I(C(\mathcal{H}),d_0)$ of $(C(\mathcal{H}),d_0)$ is trivial, that is one has*

$$H^0_I(C(\mathcal{H}),d_0)=\mathbb{K} \quad and \quad H^n_I(C(\mathcal{H}),d_0)=0$$

for $n\geqslant 1$.

Proof. The contracting homotopy K used in the proof of Proposition 3 commutes with the L_h given by (5.6) which implies the result. □

Note that the extra term ε_h in L_h which is independent of i_h plays no role here, (in fact $L_{\mathbb{1}}=I_{C(\mathcal{H})}$).

5.3 Algebraic connections

Let $\mathcal{H}$ be a Hopf algebra, Ω be a graded differential and assume that one has an operation $h\mapsto i_h$ of $\mathcal{H}$ in Ω, (i.e. that (i,Ω) is a $\mathcal{H}$-operation).

An algebraic connection on the $\mathcal{H}$-operation (i,Ω) or simply *a connection on* (i,Ω) is a homomorphism of graded algebras

$$\alpha: C(\mathcal{H})\to\Omega \tag{5.8}$$

such that

$$i_h(\alpha(\Psi))=\alpha(i_h(\Psi)) \tag{5.9}$$

and

$$L_h(\alpha(\Psi))=\alpha(L_h(\Psi)) \tag{5.10}$$

for any $\Psi \in C(\mathcal{H})$, which implies

$$i_h(\alpha(\psi)) = \psi(h) - \varepsilon(h)\psi(\mathbb{1}) \tag{5.11}$$

and

$$L_h(\alpha(\psi)) = \alpha(\psi \circ \mathrm{ad}(h)) \tag{5.12}$$

for any $h \in \mathcal{H}$ and $\psi \in \mathcal{H}^*$.

The *curvature of* α is the homogeneous linear mapping of degree 1

$$\varphi : C(\mathcal{H}) \to \Omega$$

defined by

$$\varphi(\Psi) = (d\alpha - \alpha d)(\Psi) \tag{5.13}$$

for any $\Psi \in C(\mathcal{H})$. As in the classical theory, the curvature is the obstruction for α to be a homomorphism of graded differential algebras. This implies that

$$i_h(\varphi(\psi)) = 0 \tag{5.14}$$

and

$$L_h(\varphi(\psi)) = \varphi(\psi \circ \mathrm{ad}(h)) \tag{5.15}$$

for any $h \in \mathcal{H}$ and $\psi \in \mathcal{H}^*$, and more generally

$$i_h(\varphi(\Psi)) = -\varphi(i_h(\Psi)) \tag{5.16}$$

and

$$L_h(\varphi(\Psi)) = \varphi(L_h(\Psi)) \tag{5.17}$$

for any $\Psi \in C(\mathcal{H})$, together with

$$\varphi(\mathbb{1}) = 0 \tag{5.18}$$

that is $\varphi(C^0(\mathcal{H})) = 0$. Furthermore (5.13) implies also that

$$\varphi(\Phi\Psi) = \varphi(\Phi)\alpha(\Psi) + (-1)^f \alpha(\Phi)\varphi(\Psi) \tag{5.19}$$

for $\Phi \in C^f(\mathcal{H}), \Psi \in C(\mathcal{H})$ and that one has the following version of Bianchi identity

$$d(\varphi(\Psi)) = -\varphi(d\Psi) \tag{5.20}$$

for any $\Psi \in C(\mathcal{H})$.

Notice that α induces a structure of $C(\mathcal{H})$-bimodule on Ω and that then (5.19) means that φ is an antiderivation of $C(\mathcal{H})$ into Ω, (see in §6.2).

Remark.

1. The minus sign in front of the right-hand side of (5.16) and (5.20) are connected with the Koszul convention for composition of homogeneous linear mappings between graded vector spaces.

2. Once the Koszul convention is adopted all relations (5.9), (5.10), (5.16), (5.17) appear as "commutation properties" and, by an obvious abuse of notations and by using corresponding graded commutators, can be written as

$$[\alpha, i_h] = 0,\ \ [\alpha, L_h] = 0,\ \ [\varphi, i_h] = 0,\ \ [\varphi, L_h] = 0$$

respectively while (5.13) and (5.20) can be written respectively

$$\varphi = [d, \alpha], \quad [d, \varphi] = 0$$

with the same convention.

One verifies that the identity mapping $I_{C(\mathcal{H})}$ of $C(\mathcal{H})$ onto itself is a connection α_C on the $\mathcal{H}$-operation $C(\mathcal{H})$ defined in the last section and that this connection is *flat*, that is has a vanishing curvature ($\varphi_C = 0$). This connection will be refered to as *the canonical flat connection of* $C(\mathcal{H})$. It is worth noticing here that in the case $\mathcal{H} = U(\mathfrak{g})$ where $\mathfrak{g}$ is the Lie algebra of a Lie group G, $\mathfrak{g} = \mathrm{Lie}(G)$, the Maurer-Cartan equation on G relies directly to the flatness of this connection on $C(\mathcal{H})$.

5.4 The case $\dim(\mathcal{H}) < \infty$ and the case $\dim(\mathfrak{g}) = \infty$

If $\dim(\mathcal{H}) < \infty$, then $C(\mathcal{H})$ is the tensor algebra $T(\mathcal{H}^*)$ and a connection α is completely specified by its restriction to $\mathcal{H}^*$. Thus in this case, a connection is simply a linear mapping

$$\alpha : \mathcal{H}^* \to \Omega^1 \tag{5.21}$$

satisfying (5.11) and (5.12).

Since in this case one has the isomorphism

$$\mathrm{Hom}(\mathcal{H}^*, \Omega^1) \simeq \mathcal{H} \otimes \Omega^1$$

of vector spaces, one can also say that a connection on the operation of $\mathcal{H}$ in Ω is an element A of degree 1 of $\mathcal{H}\otimes\Omega$, that is

$$A \in \mathcal{H}\otimes\Omega^1 \tag{5.22}$$

such that

$$i_h(A) = h - \varepsilon(h)\mathbb{1} \tag{5.23}$$

and

$$L_h(A) = \mathrm{ad}(h)A \tag{5.24}$$

for any $h \in \mathcal{H}$.

Given such a connection A, *its curvature* $F = F(A)$ is the element

$$F = dA + A^2 \tag{5.25}$$

of $\mathcal{H}\otimes\Omega^2$ corresponding to $\varphi \in \mathrm{Hom}(\mathcal{H}^*, \Omega^2)$.

The curvature F of A satisfies

$$i_h(F) = 0 \tag{5.26}$$

and

$$L_h(F) = \mathrm{ad}(h)F \tag{5.27}$$

for any $h \in \mathcal{H}$. This can be checked directly by using (5.25), (5.23), (5.24) and the definition of $\mathcal{H}$-operations. Applying d to (5.25) implies

$$dF + AF - FA = 0 \tag{5.28}$$

which is the Bianchi identity in the present context.

One sees that the case where $\dim(\mathcal{H}) < \infty$ looks formally close to the classical theory summarized in Section 2. However, it is worth noticing here that in Section 2 as well as in the original references [7] and [8], it is implicitely assumed that the Lie algebra $\mathfrak{g}$ is finite-dimensional. If one wants to extend the classical theory (where all graded algebras are graded commutative) to the case where $\dim(\mathfrak{g}) = \infty$, then one must change accordingly the definition of algebraic connections. Indeed, the transposed of the Lie bracket is now a

linear mapping of $\mathfrak{g}^*$ into the space $C^2_\wedge(\mathfrak{g})$ of antisymmetric bilinear forms on $\mathfrak{g}$ and therefore one has to replace $\wedge\mathfrak{g}^*$ by the graded commutative algebra

$$C_\wedge(\mathfrak{g}) = \oplus_n C^n_\wedge(\mathfrak{g})$$

where $C^n_\wedge(\mathfrak{g})$ is the vector space of completely antisymmetric n-linear forms on $\mathfrak{g}$ and one endows $C_\wedge(\mathfrak{g})$ with the Chevalley-Eilenberg differential d of cochains on $\mathfrak{g}$ with values in the trivial representation (in $\mathbb{K}$). Then, given an operation i of $\mathfrak{g}$ in the graded commutative differential algebra $\Omega = \oplus_n \Omega^n$, an algebraic connection on Ω should be defined as a homomorphism of graded commutative algebras

$$\alpha : C_\wedge(\mathfrak{g}) \to \Omega$$

satisfying conditions similar to (5.9) and (5.10) with curvature φ again defined by

$$\varphi = d \circ \alpha - \alpha \circ d$$

with obvious notations, etc.

Thus for $\dim(\mathfrak{g}) = \infty$, all definitions and formulas look completely similar to the one of §5.3, except that we are in a graded commutative context with the classical notion of operation of the Lie algebra $\mathfrak{g}$.

The definition of the Weil algebra $W(\mathfrak{g})$ of an infinite-dimensional Lie algebra $\mathfrak{g}$ must be modified accordingly and looks then closer to the definition of the Weil algebra $W(\mathcal{H})$ of a Hopf algebra $\mathcal{H}$ defined in Section 6.

5.5 The affine space of connections on a $\mathcal{H}$-operation

In the case $\dim(\mathcal{H}) < \infty$, it is clear that the set of all connections on a $\mathcal{H}$-operation (i, Ω) is an affine space. This is for instance obvious by using the formulation of the last section based on (5.22), (5.23), (5.24).

This comes from the fact that for $\dim(\mathcal{H}) < \infty$, one has $T(\mathcal{H}^*) = C(\mathcal{H})$ and that, in view of the universal property of the tensor algebra $T(\mathcal{H}^*)$,

$$\alpha \mapsto \alpha \restriction \mathcal{H}^*$$

is a bijection of the set of homomorphisms of graded algebras

$$\alpha : T(\mathcal{H}^*) \to \Omega$$

onto the set $\mathrm{Hom}(\mathcal{H}^*, \Omega^1)$ of linear mapping of $\mathcal{H}^*$ into Ω^1. Since $\mathrm{Hom}(\mathcal{H}^*, \Omega^1)$ is a vector space this implies that there is a natural structure of vector space on the set of homomorphisms of graded algebras of $C(\mathcal{H})$ into a given graded algebra Ω whenever $\mathcal{H}$ is finite-dimensional.

In general one has the inclusion

$$T(\mathcal{H}^*) \subset C(\mathcal{H})$$

of $T(\mathcal{H}^*)$ as graded subalgebra of $C(\mathcal{H})$ and this inclusion is a strict one if $\dim(\mathcal{H}) = \infty$. Thus $T(\mathcal{H}^*) = C(\mathcal{H})$ is equivalent to $\dim(\mathcal{H}) < \infty$.

Nevertheless let us show that there is always a natural structure of vector space on the set of homomorphisms of graded algebras $\alpha : C(\mathcal{H}) \to \Omega$ of $C(\mathcal{H})$ into a given graded algebra Ω. To understand what is involved here, let us analyse degree by degree the arbitrariness of such a homomorphism α.

In degree 0, there is no arbitrariness since α must be unital and $C^0(\mathcal{H}) = \mathbb{K}\mathbb{1}$. Thus the arbitrariness is contained in

$$\alpha_+ : C^+(\mathcal{H}) \to \Omega^+$$

where $C^+(\mathcal{H}) = \oplus_{n\geqslant 1} C^n(\mathcal{H})$ and $\Omega^+ = \oplus_{n\geqslant 1}\Omega^n$. In degree 1, α_1 is an arbitrary linear mapping of $\mathcal{H}^* = C^1(\mathcal{H})$ into Ω^1. In degree 2, α_2 is fixed on $(C^1(\mathcal{H}))^2 \subset C^2(\mathcal{H})$ in terms of α_1 by the homomorphism property but remains arbitrary on a supplementary $V^2(\mathcal{H}) \simeq C^2(\mathcal{H})/(C^1(\mathcal{H}))^2$ to $(C^1(\mathcal{H}))^2$ in $C^2(\mathcal{H})$. More generally consider the quotient

$$V(\mathcal{H}) = C^+(\mathcal{H})/(C^+(\mathcal{H}))^2$$

of $C^+(\mathcal{H})$ by $(C^+(\mathcal{H}))^2$. The product of $C^+(\mathcal{H})$ induces the trivial zero product on $V(\mathcal{H})$ which is now just a graded vector space. By chosing a supplementary of $(C^+(\mathcal{H}))^2$ in $C^+(\mathcal{H})$, one sees that the arbitrariness in the definition of a homomorphism α of graded algebras of $C(\mathcal{H})$ into Ω is just a homogeneous linear mapping

$$\bar{\alpha} : V(\mathcal{H}) \to \Omega$$

of degree 0, $(V(\mathcal{H}) = \oplus_{n\geqslant 1} V^n(\mathcal{H}))$. The set of these linear mappings $\bar{\alpha}$ is a vector space. Notice that $\bar{\alpha}_1 = \alpha_1$ since $V^1(\mathcal{H}) = C^1(\mathcal{H})$ and that $V^n(\mathcal{H}) = 0$

for $n \geqslant 2$ whenever $\dim(\mathcal{H}) < \infty$.

In order that α be a connection it should satisfy (5.9) which is inhomogeneous in terms of $\bar{\alpha}$ (see (5.11)) but becomes homogeneous in $\bar{\alpha} - \bar{\alpha}'$ for two connections α and α'. The other condition (5.10) being homogeneous, one sees that the connections form an affine space.

6 The universal $\mathcal{H}$-operation with connection

In this section $\mathcal{H}$ is a fixed Hopf algebra and we define a noncommutative version of the Weil algebra, the Weil algebra $W(\mathcal{H})$ of the Hopf algebra $\mathcal{H}$.

6.1 The category of $\mathcal{H}$-operations with connections

Let f be a morphism of $\mathcal{H}$-operation from (i, Ω) to (i', Ω') and let

$$\alpha : C(\mathcal{H}) \to \Omega$$

be a connection on (i, Ω), then the image $f \circ \alpha$ of α by f

$$f \circ \alpha : C(\mathcal{H}) \to \Omega'$$

is a connection on (i', Ω') which will be denoted by $f(\alpha)$.

A $\mathcal{H}$-operation with connection (i, Ω, α) is an $\mathcal{H}$-operation (i, Ω) equipped with a connection α. Given two $\mathcal{H}$-operations with connections (i, Ω, α) and (i', Ω', α'), *a morphism of $\mathcal{H}$-operation with connection* from (i, Ω, α) to (i', Ω', α') is a morphism of $\mathcal{H}$-operation f from (i, Ω) to (i', Ω') such that $\alpha' = f(\alpha)$. This defines the category of $\mathcal{H}$-operations with connections.

It turns out that this category of $\mathcal{H}$-operations with connections has a universal initial object $W(\mathcal{H})$ which is the appropriate generalization of the Weil algebra in our context and which will be described in this section.

We first describe in the next subsection the construction of the universal differential calculus over a graded algebra.

6.2 Differential envelopes of graded algebras

Let $\mathcal{A} = \oplus_n \mathcal{A}^n$ be a graded algebra ($\mathbb{N}$-graded, unital, associative) with product

$$(x, y) \mapsto m(x \otimes y) = xy$$

for $x, y \in \mathcal{A}$. Let $\mathcal{M}$ be a $\mathcal{A}$-bimodule, an *antiderivation* of $\mathcal{A}$ into $\mathcal{M}$ is a linear mapping

$$\delta : \mathcal{A} \to \mathcal{M}$$

such that one has

$$\delta(ab) = \delta(a)b + (-1)^r a\delta(b) \tag{6.1}$$

for $a \in \mathcal{A}^r$ and $b \in \mathcal{A}$. Following [17] let us define the twisted product

$$\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$$

on $\mathcal{A}$ by setting

$$\mu(a \otimes b) = (-1)^s ab \tag{6.2}$$

for $a \in \mathcal{A}$ and $b \in \mathcal{A}^s$. There is a structure of $\mathcal{A}$-bimodule on $\mathcal{A} \otimes \mathcal{A}$ given by setting

$$x(a \otimes b) = xa \otimes b, \quad (a \otimes b)y = a \otimes by$$

for $x, y, a, b \in \mathcal{A}$ and the kernel J of μ is a sub-bimodule of $\mathcal{A} \otimes \mathcal{A}$. One verifies that one defines an antiderivation $d : \mathcal{A} \to J$ of $\mathcal{A}$ into the $\mathcal{A}$-bimodule J by setting

$$d(x) = \mathbb{1} \otimes x - (-1)^n x \otimes \mathbb{1} \tag{6.3}$$

for $x \in \mathcal{A}^n$. This antiderivation $d : \mathcal{A} \to J$ is characterized (up to isomorphisms) by the following universal property.

Proposition 5. *Assume that $\delta : \mathcal{A} \to \mathcal{M}$ is an antiderivation of $\mathcal{A}$ into an $\mathcal{A}$-bimodule $\mathcal{M}$. Then there is a unique homomorphism of $\mathcal{A}$-bimodules $i_\delta : J \to \mathcal{M}$ of J into $\mathcal{M}$ such that $\delta = i_\delta \circ d$.*

Thus this construction of [17] gives the counterpart for antiderivations of the classical construction [9] of the universal derivations (see also [3]), furthermore the proof of Proposition 5 is the same as the proof of the corresponding proposition for derivations in [9], [3]. A key remark for the proof is that as left $\mathcal{A}$-module, one has the isomorphisms

$$J \simeq \mathcal{A}d\mathcal{A} \simeq \mathcal{A} \otimes d\mathcal{A} \simeq \mathcal{A} \otimes (\mathcal{A}/\mathbb{K}\mathbb{1})$$

while the right $\mathcal{A}$-module structure is obtained from above by using the graded Leibniz rule (6.1).

The above results apply as well for $\mathbb{Z}$-graded or $\mathbb{Z}_2$-graded algebras, in fact the paper [17] is written in the $\mathbb{Z}_2$-graded context. Nevertheless for the following $\mathcal{A}$ is assumed to be $\mathbb{N}$-graded.

We now introduce a graduation on $J \subset \mathcal{A} \otimes \mathcal{A}$ by setting $J = \oplus_{n\geqslant 0} J^{n+1}$ with

$$J^{n+1} \subseteqq \oplus_{r+s=n} \mathcal{A}^r \otimes \mathcal{A}^s$$

for $n \in \mathbb{N}$. Endowed with this graduation J becomes a graded $\mathcal{A}$-bimodule which will be denoted by $\Omega^1_{gr}(\mathcal{A})$. Thus

$$d : \mathcal{A} \to \Omega^1_{gr}(\mathcal{A})$$

is then a graded derivation of degree 1 of $\mathcal{A}$ into $\Omega^1_{gr}(\mathcal{A})$. Note that the kernel of d is $\mathbb{K}\mathbb{1}$ so that

$$d\mathcal{A} \simeq (\mathcal{A}/\mathbb{K}\mathbb{1})^{\bullet+1}$$

i.e. $d\mathcal{A}$ is $\mathcal{A}/\mathbb{K}\mathbb{1}$ with a shift $+1$ in graduation so $\Omega^1_{gr}(\mathcal{A}) \simeq \mathcal{A} \otimes (\mathcal{A}/\mathbb{K}\mathbb{1})^{\bullet+1}$.

Let us now define the graded algebra $\Omega_{gr}(\mathcal{A}) = \oplus_n \Omega^n_{gr}(\mathcal{A})$ to be the tensor algebra over $\mathcal{A}$ of the bimodule $\Omega^1_{gr}(\mathcal{A})$ endowed with the unique graduation which induces on $\mathcal{A} = \Omega^0_{gr}(\mathcal{A})$ and on $\Omega^1_{gr}(\mathcal{A})$ their original graduation.

The graded derivation d of $\mathcal{A}$ into $\Omega^1_{gr}(\mathcal{A})$ has a unique extension as a differential on $\Omega_{gr}(\mathcal{A})$, i.e. as a graded derivation of degree 1 of $\Omega_{gr}(\mathcal{A})$, again denoted by d, satisfying $d^2 = 0$. Endowed with this differential, $\Omega_{gr}(\mathcal{A})$ is a graded differential algebra which is characterized, up to an isomorphism, by the following universal property which is a graded counterpart of the universal property of the usual universal differential calculus $\Omega(\mathcal{A})$ over a non graded algebra $\mathcal{A}$ [10], [11], [27], [28].

Theorem 6. *Any homomorphism of graded algebras*

$$\alpha : \mathcal{A} \to \Omega$$

of $\mathcal{A}$ into a graded differential algebra Ω has a unique extension as homomorphism

$$\Omega_{gr}(\alpha) : \Omega_{gr}(\mathcal{A}) \to \Omega$$

of graded differential algebras.

The proof is completely similar to the proof of the "ungraded counterpart". Notice that this theorem gives a great flexibility for its applications in the graded algebra context (in the above sense).
In spite of the fact that its proof is easy this result is new to our knowledge.

6.3 The graded differential algebra $W(\mathcal{H})$

Let (i, Ω, α) be a $\mathcal{H}$-operation with connection. Since then α is in particular a homomorphism of graded algebra of $C(\mathcal{H})$ into Ω, it is natural to introduce the graded differential algebra

$$W(\mathcal{H}) = \Omega_{gr}(C(\mathcal{H})) \tag{6.4}$$

with the notations of the last subsection.

Theorem 6 and (6.4) have the following corollary.

Corollary 7. *Let (i, Ω, α) be a $\mathcal{H}$-operation with connection, then*

$$\alpha : C(\mathcal{H}) \to \Omega$$

has a unique extension

$$W(\alpha) : W(\mathcal{H}) \to \Omega$$

as homomorphism of graded differential algebras.

6.4 The $\mathcal{H}$-operation with connection $W(\mathcal{H})$

Let us denote by

$$\alpha_W : C(\mathcal{H}) \to W(\mathcal{H})$$

the canonical injection of $C(\mathcal{H})$ into $W(\mathcal{H})$ as graded subalgebra and by

$$\varphi_W = d \circ \alpha_W - \alpha_W \circ d : C(\mathcal{H}) \to W(\mathcal{H})$$

the corresponding obstruction for α_W to be a homomorphism of graded differential algebra. The homogeneous linear mapping φ_W satisfied the relations (5.18), (5.19) and (5.20) that is $\varphi_W(\mathbb{1}) = 0$

$$\varphi_W(\Phi\Psi) = \varphi_W(\Psi)\alpha_W(\Psi) + (-1)^f \alpha_W(\Phi)\varphi_W(\Psi)$$

and

$$d\varphi_W(\Psi) = -\varphi_W(d\Psi)$$

for $\Phi \in C^f(\mathcal{H})$, $\Psi \in C(\mathcal{H})$.

Proposition 8. *There is a unique operation $h \mapsto i_h$ of the Hopf algebra $\mathcal{H}$ in the graded differential algebra $W(\mathcal{H})$ for which α_W is an algebraic connection*

Proof. In view of the definitions of Subsection 5.3, if α_W is a connection, its curvature is given by φ_W. Thus one should have

$$i_h(\alpha_W(\Psi)) = \alpha_W(i_h(\Psi))$$

and

$$i_h(\varphi_W(\Psi)) = \varphi_W(i_h(\Psi))$$

for $h \in \mathcal{H}$, $\Psi \in C(\mathcal{H})$. This fixes the i_h on $\alpha_W(C(\mathcal{H})) \simeq C(\mathcal{H})$ and on $\varphi_W(C(\mathcal{H}))$. One verifies that the relations $[\alpha_W, L_h] = 0$ and $[\varphi_W, L_h] = 0$ are satisfied and then the relation (4.3) fixes i_h on $W(\mathcal{H})$. One then verifies that so defined $h \mapsto i_h$ is an operation of $\mathcal{H}$ in $W(\mathcal{H})$. $\square$

By combining Corollary 7 and Proposition 8 one arrives at the following theorem.

Theorem 9. *Let Ω be a $\mathcal{H}$-operation with connection, then there is a unique morphism of $\mathcal{H}$-operation with connection from $W(\mathcal{H})$ to Ω.*

In other words $W(\mathcal{H})$ is a universal initial object in the category of $\mathcal{H}$-operations with connections. As such it is unique up to isomorphism.

The graded differential algebra $W(\mathcal{H})$ endowed with the structure described above will be refered to as *the Weil algebra of the Hopf algebra $\mathcal{H}$.*

It is clear that $W(\mathcal{H})$ plays in the present setting the same role as the Weil algebra $W(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ in the classical theory and it is worth noticing here that the role of $\wedge\mathfrak{g}^*$ in the classical theory is played in our noncommutative framework by $C(\mathcal{H})$.

One has canonically

$$W(\mathcal{H}) = C(\mathcal{H}) \oplus W_\varphi(\mathcal{H}) \tag{6.5}$$

where $W_\varphi(\mathcal{H})$ is the two-sided ideal of $W(\mathcal{H})$ generated by $\varphi_W(C(\mathcal{H}))$. This ideal is in fact a graded differential ideal of $W(\mathcal{H})$ so the corresponding canonical surjective homomorphism

$$\rho : W(\mathcal{H}) \to C(\mathcal{H}) \tag{6.6}$$

is a homomorphism of graded differential algebras. It is easy to see that it is the morphism of $\mathcal{H}$-operation with connection of Theorem (9) for $\Omega = C(\mathcal{H})$ where the graded differential algebra $C(\mathcal{H})$ is endowed with its structure of $\mathcal{H}$-operation with (flat) connection described in §5.2 and §5.3. One has

$$\rho \circ \alpha_W = I_{C(\mathcal{H})} \tag{6.7}$$

which means that the connection of $W(\mathcal{H})$ is a section of ρ.

6.5 Cohomology and invariant cohomology of $W(\mathcal{H})$

The cohomology and the invariant cohomology of $W(\mathcal{H})$ are given by the following theorem.

Theorem 10. *The cohomology $H(W(\mathcal{H}))$ and the invariant cohomology $H_I(W(\mathcal{H}))$ of $W(\mathcal{H})$ are trivial, that is one has*

$$H^n(W(\mathcal{H})) = H^n_I(W(\mathcal{H})) = 0$$

for $n \geqslant 1$ while $H^0(W(\mathcal{H}))$ and $H^0_I(W(\mathcal{H}))$ identify to the ground field $\mathbb{K}$.

Proof. By its very definition, $W(\mathcal{H})$ is the tensor algebra over $C(\mathcal{H})$ of the $C(\mathcal{H})$-bimodule $\Omega^1_{gr}(C(\mathcal{H}))$

$$W(\mathcal{H}) = T_{C(\mathcal{H})}(\Omega^1_{gr}(C(\mathcal{H}))),$$

therefore, there is a unique antiderivation K of $W(\mathcal{H})$ which is such that

$$K \circ \alpha_W = 0 \tag{6.8}$$

and which satisfies

$$K \circ d \circ \alpha_W = \deg \circ \alpha_W \tag{6.9}$$

where deg is the degree, (since $\alpha_W(C(\mathcal{H}))$ and $d(\alpha_W(C(\mathcal{H})))$ generate $W(\mathcal{H})$). In fact deg is a derivation of $W(\mathcal{H})$ into itself as well as a derivation of $C(\mathcal{H})$ into itself and one has $\deg \circ \alpha_W = \alpha_W \circ \deg$.

Notice that $\alpha_W(C(\mathcal{H})$ and $\varphi_W(C(\mathcal{H}))$ generate as well $W(\mathcal{H})$ and that (6.9) is equivalent to

$$K \circ \varphi_W = \deg \circ \alpha_W \tag{6.10}$$

in view of the definition of φ_W. This implies that one has

$$K \circ L_h = L_h \circ K \tag{6.11}$$

and

$$(K \circ d + d \circ K) \circ \alpha_W = \alpha_W \circ \deg = \deg \circ \alpha_W$$

$$(K \circ d + d \circ K) \circ d \circ \alpha_W = d \circ \alpha_W \circ \deg = (\deg - I) \circ d \circ \alpha_W$$

on $C(\mathcal{H})$ which implies $H^n(W(\mathcal{H})) = 0$ for $n \geqslant 1$ and by using (6.11), $H^n_I(W(\mathcal{H})) = 0$ for $n \geqslant 1$. On the other hand $H^0(W(\mathcal{H})) = H^0_I(W(\mathcal{H})) = \mathbb{K}$ is obvious. □

7 Conclusion

This paper is the first part of a work on the noncommutative generalization of the notion of Cartan operation and of the Weil algebra. In this first part we have set up the general formulation of this noncommutative version. The second part will be devoted to the description in this context of the noncommutative version of the Weil homomorphism and of the noncommutative version of the Cartan map.

Let us now explain why we did not mention the axiom $(i_X)^2 = 0$ $(\forall X \in \mathfrak{g})$ for the operation of a Lie algebra $\mathfrak{g}$ in a graded differential algebra. Firstly, $(i_X)^2$ is (for $X \in \mathfrak{g}$) a derivation of degree -2 of Ω which implies that it vanishes on the graded subalgebra of Ω generated by the elements of degrees 0 and 1, (remembering that Ω is positively graded by assuption). Secondly the axiom 2.3 (Cartan relation) implies that one has

$$[(i_X)^2, d] = 0$$

for any $X \in \mathfrak{g}$. Thus one has $(i_X)^2 = 0$ on the graded differential subalgebra $\Omega_{(1)}$ of Ω generated (as graded differential algebra) by the elements of degrees 0 and 1. In all cases of interest one has $\Omega_{(1)} = \Omega$, that is Ω is generated as graded differential algebra by $\Omega^0 \oplus \Omega^1$ which implies $(i_X)^2 = 0$ for $X \in \mathfrak{g}$. Thus one needs not the axiom $(i_X)^2 = 0$ which plays no role otherwise.

Finally let us say some words on the relation with the construction of [18]. In the interesting paper [18] there is a definition of the Weil algebra of a coalgebra which leads of course to a definition of a Weil algebra of a Hopf algebra

$\mathcal{H}$. However the corresponding Weil algebra is generated by $\mathcal{H}$ instead of $\mathcal{H}^*$ as our $W(\mathcal{H})$. Thus in spite of some similarities, it is a different object which is considered in [18]. In particular our correspondence $\mathcal{H} \mapsto W(\mathcal{H})$ has the same variance as the classical correspondence $\mathfrak{g} \mapsto W(\mathfrak{g})$ ($W(\mathfrak{g})$ is generated by $\mathfrak{g}^*$) while the Weil algebra of [18] has an opposite variance. Nevertheless, it is clear that for the noncommutative Weil homomorphism, we shall use results of [18] as well as those of [14], [15] and [16].

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References

[1] A. Alekseev and E. Meinrenken. The non-commutative Weil algebra. *Inv. Math.*, 139:135–172, 2000.

[2] P. Aschieri and L. Castellani. An introduction to noncommutative differential geometry on quantum groups. *Int. J. Mod. Phys.*, A8:1667–1706, 1993.

[3] N. Bourbaki. *Algèbre I, Chapitre III.* Hermann, 1970.

[4] S. Brain and G. Landi. Differential and twistor geometry of the quantum Hopf fibration. ArXiv:1103.0419.

[5] T. Brzeziński and S. Majid. Quantum group gauge theory on quantum spaces. *Commun. Math. Phys.*, 157:591–638, 1993.

[6] T. Brzeziński and S. Majid. Erratum. *Commun. Math. Phys.*, 167:235, 1995.

[7] H. Cartan. Notion d'algèbre différentielle; application aux groupes de Lie et aux variétés où opère un groupe de Lie. In *Colloque de Topologie, Bruxelles, 1950*, pages 15–27. Masson, 1951.

[8] H. Cartan. La trangression dans un groupe de Lie et dans un espace fibré principal. In *Colloque de Topologie, Bruxelles, 1950*, pages 57–71. Masson, 1951.

[9] H. Cartan and S. Eilenberg. *Homological algebra*. Princeton University Press, 1973.

[10] A. Connes. Noncommutative differential geometry. Part I: The Chern character in K-homology; Part II: de Rham homology and noncommutative algebra. *Preprints IHES M/82/53; M83/19*, 1982/83.

[11] A. Connes. *Non-commutative geometry*. Academic Press, 1994.

[12] A. Connes and M. Dubois-Violette. Noncommutative finite-dimensional manifolds. I.Spherical manifolds and related examples. *Commun. Math. Phys.*, 230:539–579, 2002.

[13] A. Connes and G. Landi. Noncommutative manifolds, the instanton algebra and isospectral deformations. *Commun. Math. Phys.*, 221:141–159, 2001.

[14] A. Connes and H. Moscovici. Hopf algebras, cyclic cohomology and the transverse index theorem. *Commun. Math. Phys.*, 198:199–246, 1998.

[15] A. Connes and H. Moscovici. Cyclic cohomology and Hopf algebras. *Lett. Math. Phys.*, 48:97–108, 1999.

[16] A. Connes and H. Moscovici. Cyclic cohomology and Hopf algebra symmetry. *Lett. Math. Phys.*, 53:1–28, 2000.

[17] R. Coquereaux and D. Kastler. Remarks on the differential envelopes of associative algebras. *Pacific J. Math.*, 137:245–263, 1989.

[18] M. Crainic. Cyclic cohomology of Hopf algebras. *Journal of Pure and Applied Algebra*, 166:29–66, 2002.

[19] M. Dubois-Violette. The Weil-B.R.S. algebra of a Lie algebra and the anomalous terms in gauge theory. *J. Geom. Phys.*, 3:525–565, 1987.

[20] M. Dubois-Violette. Dérivations et calcul différentiel non-commutatif. *C.R. Acad. Sci. Paris*, 307:403–408, 1988.

[21] M. Dubois-Violette. Lectures on graded differential algebras and noncommutative geometry. In Y. Maeda and al., editors, *Noncommutative Differential Geometry and Its Applications to Physics*, pages 245–306. Shonan, Japan, 1999, Kluwer Academic Publishers, 2001.

[22] M. Dubois-Violette, M. Henneaux, M. Talon, and C.M. Viallet. General solution of the consistency equation. *Phys. Lett.*, B289:361–367, 1992.

[23] M. Dubois-Violette and G. Landi. Lie prealgebras. In A. Connes and al., editors, *Noncommutative Geometry and Global Analysis*, volume 546 of *Contemporary Mathematics*, pages 115–135. American Mathematical Society, 2011.

[24] M. Dubois-Violette, M. Talon, and C.M. Viallet. B.R.S. algebras. analysis of consistency equations in gauge theory. *Commun. Math. Phys*, 102:105 –122, 1985.

[25] W. Greub, S. Halperin, and R. Vanstone. *Connections, curvature, and cohomology*, volume III. Academic Press, 1976.

[26] V. Guillemin and S. Sternberg. *Supersymmetry and equivariant de Rham theory*. Springer Verlag, 1999.

[27] M. Karoubi. Homologie cyclique des groupes et algèbres. *C.R. Acad. Sci. Paris, Série I*, 297:381–384, 1983.

[28] M. Karoubi. *Homologie cyclique et K-théorie*, volume 149 of *Astérique*. Société Mathématique de France, 1987.

[29] A. Klimyk and K. Schmüdgen. *Quantum groups and their representations*. Springer, 1997.

[30] M. Kobayashi and K. Nomizu. *Foundations of differential geometry*, volume 1. Interscience publishers, 1963.

[31] G. Landi and W.D. van Suijlekom. Principal fibrations from noncommutative spheres. *Commun. Math. Phys.*, 260:203–225, 2005.

[32] G. Landi and W.D. van Suijlekom. Noncommutative instantons in Tehran. In M. Khalkhali and al., editors, *An invitation in noncommutative geometry*, pages 275–353, Hackensack, N.J., 2008. World Scientific Pub.

[33] J.L. Loday. *Cyclic homology*, volume 301 of *Grundlehren der mathematischen Wissenschaften*. Springer Verlag, New York, 1992.

[34] V. Mathai and D. Quillen. Thom classes superconnections and equivariant differential forms. *Topology*, 25:85–106, 1986.

[35] P. Schauenburg. Hopf-Galois and bi-Galois extensions. In G. Janelidze and al., editors, *Galois theory, Hopf algebras and semiabelian categories*, volume 43 of *Fields Institute Communications*, pages 469–515. American Mathematical Society, 2004.

[36] D. Sullivan. Infinitesimal computations in topology. *Pub. IHES*, 47:269–331, 1977.

[37] S.L. Woronowicz. Twisted SU(2) group. an example of noncommutative differential calculus. *Publ. RIMS, Kyoto Univ.*, 23:117–181, 1987.

[38] S.L. Woronowicz. Differential calculus on compact matrix pseudogroups (quantum groups). *Commun. Math. Phys.*, 122:129–170, 1989.